

Ordered Phases of the Anisotropic Kagome Lattice Antiferromagnet in a Field

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The antiferromagnetic Heisenberg model on an anisotropic kagome lattice may be a good minimal model for real magnetic systems as well as a limit from which the isotropic case can be better understood. We therefore study the nearest-neighbor Heisenberg antiferromagnet on an anisotropic kagome lattice in a magnetic field. Such a system should be well described by weakly interacting spin chains, and we motivate a general form for the interaction by symmetry considerations and by perturbatively projecting out the inter-chain spins. In the spin $1/2$ case, we find that the system exhibits a quantum phase transition from a ferrimagnetic ordered state to an XY ordered state as the field is increased. Finally, we discuss the appearance of magnetization plateaux in the ferrimagnetic phase.

I. INTRODUCTION

Frustrated antiferromagnets are of considerable interest because frustrating interactions lead to strong fluctuations both of classical (thermal) and quantum mechanical origin. Unfortunately, even the simplest model Hamiltonians for such materials are often not easily analyzable by conventional theoretical techniques. An outstanding example is the nearest-neighbor quantum Heisenberg antiferromagnet on the kagome lattice. This structure is frustrated to a particularly high degree, and the extensive classical ground state degeneracy may be identified as a mechanism for strongly enhanced fluctuations. This strongly limits the usefulness of the standard semi-classical spin wave technique. The behavior of the spin $S = 1/2$ case in zero field is particularly puzzling, with different numerical approaches yielding contradictory and/or puzzling results^{1,2,3,4,5,6,7}. Analytical studies have not been any more illuminating as they necessarily involve numerous approximations that often lower the symmetry of the problem substantially^{8,9,10,11,12,13}. However, it should be noted that recent experimental work on $\text{ZnCu}_3(\text{OH})_6\text{Cl}_2$ seems to favor a gapless, magnetically disordered ground state^{14,15}.

One window into the behavior of frustrated antiferromagnets is their *magnetization process*, i.e. the curve of magnetization versus applied field. Variation of the magnetization over the full range from zero to saturation opens another dimension of phase space in which to explore the phase diagram of these systems, and perhaps find physical and analytical insights. A qualitative feature to be understood is the occurrence of plateaux with quantized magnetization. The structure of these plateaux reveal some aspects of the correlations of the system. For the low-spin kagome lattice, there are suggestions of a plateau with $M = 1/3M_s$ (M_s is the saturation magnetization).^{6,16,17,18,19,20} This is a very natural structure for the kagome lattice, and can be understood as a state with two parallel and one anti-parallel spins per triangle. One experiment, however, suggests a plateau with $M = 1/2M_s$, which is not an obvious state²¹.

In this work, we consider an antiferromagnet on the *anisotropic* kagome lattice illustrated in fig. 1, which may

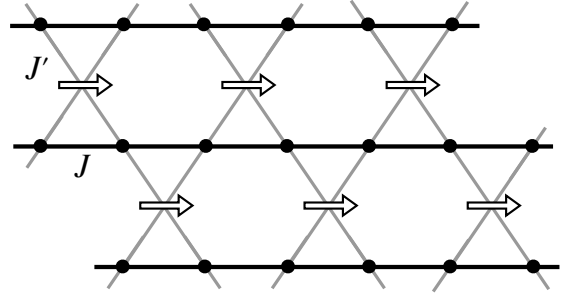


FIG. 1: The anisotropic kagome lattice with antiferromagnetic coupling J between nearest-neighbor chain spins (S spins) and coupling J' between S spins and inter-chain spins (I spins). We make the approximation that the I spins are fully polarized by the external magnetic field.

be viewed as a set of spin chains (whose spins we call S spins) coupled together through intermediate spins (I spins). The anisotropy could stem from an inhomogeneity in the spin interactions, or from an actual spatial anisotropy. A well known example of such a system is volborthite, in which spin $1/2$ moments residing on copper atoms form a kagome network that is anisotropic due to differing superexchange bond angles.²² Recently, there has been considerable progress in developing techniques to analyze such systems of quantum spin chains weakly coupled together by frustrating interactions. Such methods have the advantage that they can apply directly to low spin and full Heisenberg ($SU(2)$) symmetry, and fully include the effects of quantum fluctuations. Here we determine the physics of the magnetization process in such a limit of the kagome lattice.

We assume that the spins interact via a nearest neighbor Heisenberg interaction, and we take the magnetic field to be applied in the $+z$ direction. Taking the convention that lowercase indices only sum over sites on the spin chains (S -spin sites) while uppercase indices only sum over intermediate I -spin sites, we decompose the

Hamiltonian as

$$\begin{aligned}
H &= H_0 + H' \\
H_0 &= J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + J' \sum_{\langle Ai \rangle} I_A^z S_i^z - h \sum_i S_i^z - h \sum_A I_A^z \\
H' &= \frac{J'}{2} \sum_{\langle Ai \rangle} S_i^+ I_A^- + S_i^- I_A^+ .
\end{aligned}$$

This decomposition is convenient in the limit we focus upon, namely $J' \ll J$ and $J' \ll h$. Note that we need not assume h/J is either large or small, and so can explore (almost) the full magnetization process. The region around $h = 0$ is excluded however by $J' \ll h$. We leave this more challenging limit for future work. We have chosen to incorporate the J' term coupling the z-components of chain spins to inter-chain spins into H_0 , which is convenient since that term commutes with the large field (h) term. This however means that the expansions in powers of H' and in J' do not strictly coincide, since the latter arise from two sources.

Because J' is much smaller than the applied field, the system may be described by a low-energy effective theory involving only states in which the I spins are fully polarized. In the limit $J'/J \rightarrow 0$, one expects that this low-energy effective theory will be a set of independent Luttinger liquids describing each chain (this will always be the case for $S = 1/2$ and will also hold for $S = 1$ as long as the external field is larger than the zero-field gap to magnetic excitations). For very small J' , we continue to assume that the Luttinger liquid description is accurate, but there must also be weak inter-chain interactions. While the inter-chain interactions may be calculated, in principle, by symmetry considerations alone, we find it convenient to use an approach combining perturbation theory and symmetry arguments that allows the couplings to be estimated by hand.

For the case of a spin 1 system in a large enough field, we find that only one relevant interaction term emerges in an RG analysis of the general inter-chain coupling. Further analysis predicts that the resulting long-wavelength theory favors an XY antiferromagnetically ordered phase at zero temperature.

For the case of a spin 1/2 system, however, an RG analysis of the inter-chain coupling reveals two relevant interaction terms. For large applied field, one interaction dominates the other in the long wavelength limit, leading to the same XY antiferromagnetically ordered phase as in the spin 1 case. However, for small applied field, the other relevant interaction dominates, leading to a ferrimagnetically ordered phase. Finally, we reconcile the discrete translational symmetry of the microscopic lattice with our continuum theory and show that magnetization plateaux may appear in the ferrimagnetic phase close to the transition.

II. PERTURBATIVE CALCULATION OF EFFECTIVE HAMILTONIAN

Since we consider the limit $J'/J \ll 1$, the low energy states are those in which the I spins are completely polarized by the external field, and so we work with an effective Hamiltonian projected into the space of such states. We approximately calculate this effective Hamiltonian as follows.

A. Elimination of inter-chain spins

Let P be the projection operator onto the subspace of all states with the I spins fully polarized, and let $Q = 1 - P$. Furthermore, for a general state $|\psi\rangle$ let $|\psi_\uparrow\rangle = P|\psi\rangle$ and $|\psi_Q\rangle = Q|\psi\rangle$ so that $|\psi\rangle = |\psi_\uparrow\rangle + |\psi_Q\rangle$.

Acting on the eigenvalue equation $H|\psi\rangle = E|\psi\rangle$ with P and then with Q gives the equations

$$H_0|\psi_\uparrow\rangle + PH'|\psi_Q\rangle = E|\psi_\uparrow\rangle \quad (1)$$

$$H'|\psi_\uparrow\rangle + H_0|\psi_Q\rangle + QH'|\psi_Q\rangle = E|\psi_Q\rangle . \quad (2)$$

Letting $R = (E - H_0)^{-1}$, (2) may be written as

$$|\psi_Q\rangle = RH'|\psi_\uparrow\rangle + RQH'|\psi_Q\rangle$$

and then iterated to give

$$|\psi_Q\rangle = \frac{1}{1 - RQH'} RH'|\psi_\uparrow\rangle . \quad (3)$$

Finally, upon replacing $|\psi_Q\rangle$ in (1) by the expression in (3), we obtain an equation for $|\psi_\uparrow\rangle$ alone:

$$\left[H_0 + PH' \frac{1}{1 - RQH'} RH' \right] |\psi_\uparrow\rangle \stackrel{\text{def}}{=} H_{\text{eff}} |\psi_\uparrow\rangle = E |\psi_\uparrow\rangle . \quad (4)$$

Note, however, that Eq.(4) is not really a linear eigenvalue equation, since H_{eff} itself depends on E through R .

B. Second Order Approximation

We would like to use the assumption that $J' \ll J$ and $J' \ll h$ to simplify the expression for H_{eff} and find an approximate eigenvalue equation for $|\psi_\uparrow\rangle$. First, we make the generalization

$$H \rightarrow H_\alpha = H_0 + \alpha H'$$

in order to more conveniently keep track of powers of H' . Henceforth, take $|\psi\rangle$ to be the ground state of H_α . The ground state energy admits an expansion

$$E = E_0 + E_1 + E_2 + \dots$$

where $E_n \propto \alpha^n$ and E_0 is the ground state energy of H_0 . H_{eff} may then be expanded as

$$H_{\text{eff}} = H_0 + H_2 + O(\alpha^3),$$

with

$$H_2 = PH'R_0H', \quad (5)$$

and $R_0 = (E_0 - H_0)^{-1}$.

The second term in the expansion of H_{eff} may be simplified by use of the fact that we are restricting it to act only on states in the image of P . One finds

$$H_2|\psi_\uparrow\rangle = \frac{(\alpha J')^2 S}{2} \sum_{\langle Aik \rangle} S_k^- \frac{1}{(E_0 - H_0) - h + J' \sum_{j \text{ nn} A} S_j^z} S_i^+ |\psi_\uparrow\rangle.$$

Within the P subspace, we may drop the ket and write

$$H_2 = \sum_A \mathcal{H}_2^A, \quad (6)$$

with

$$\mathcal{H}_2^A = \frac{(\alpha J')^2 S}{2} \sum_{\langle ik; A \rangle} S_k^- \frac{1}{(E_0 - H_0) - h + J' \sum_{j \text{ nn} A} S_j^z} S_i^+, \quad (7)$$

where i, k are summed over the neighbors of A . Note that, because of the J' dependence of the denominator in Eq.(7), \mathcal{H}_2^A contains non-trivial terms at all orders of J' greater than or equal to $O[(J')^2]$. This is convenient because it allows us to obtain some $O[(J')^4]$ contributions by an only $O(\alpha^2)$ calculation.

III. CONTINUUM LIMIT AND SYMMETRY CONSIDERATIONS

It is well-known that the low energy physics of an isolated antiferromagnetic Heisenberg chain in a uniform applied field is described by Luttinger liquid theory, whenever the field range is such that the system remains gapless. In the limit $J' \ll J, h$, which we consider, all influences of the inter-chain spins upon the spin chains indeed occur at low energies, and so can be considered in this framework.

Luttinger liquid theory consists of replacing the Hamiltonian of each spin chain by that of a free boson field (and corrections that can be analyzed perturbatively):

$$H_0 \rightarrow \sum_n H_n^{(\text{LL})}, \quad (8)$$

where

$$H_n^{(\text{LL})} = \frac{v}{2\pi} \sum_n \int_x \left[\frac{1}{g} (\partial_x \varphi_n)^2 + g (\partial_x \theta_n)^2 \right] \quad (9)$$

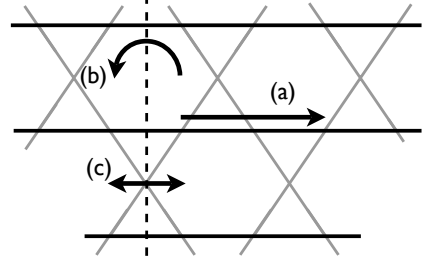


FIG. 2: The generating symmetries of the space group cmm of the anisotropic kagome lattice: (a) translations, (b) rotations by π about the hexagon centers, and (c) reflections through a vertical line passing through the hexagon centers.

is the Hamiltonian of a Luttinger liquid on chain n . Here the “spin velocity” v and the Luttinger parameter g are known functions of h/J and S . The variables θ_n and φ_n are dual boson fields living on the n^{th} chain and satisfying $[\varphi_n(x), \theta_m(y)] = -i\pi\delta_{nm}\varphi(x-y)$.

The physical interpretation of the phase fields is understood from their relation to the microscopic spin operators:

$$S_{j,n}^\pm \sim (-1)^j e^{\pm i\theta_n(x_j)} [b_0 + b_1 \cos(2\varphi_n(x_j) + Qx_j)] \quad (10)$$

$$S_{j,n}^z \sim m_{\text{ch}} + \frac{1}{\pi} \partial_x \varphi_n(x_j) + a_1 \cos(2\varphi_n(x_j) + Qx_j). \quad (11)$$

The coefficients a_1 , b_0 and b_1 depend on the chain magnetization per site m_{ch} and, for $S = 1/2$, the wavevector Q is given by $Q = 2\pi(m_{\text{ch}} + 1/2)$.

From the above considerations, we expect that H_2 (and higher order corrections) can also be expressed within the continuum (conformal) field theory. Formally, one may expand any local Hamiltonian in terms of the scaling operators of the decoupled fixed point theory. Specifically, the density can be written as

$$\mathcal{H}_2^A = \sum_\sigma \lambda_\sigma \mathcal{O}_{n_A, n_A+1}^\sigma(x_A), \quad (12)$$

where $\mathcal{O}_{n,n+1}^\sigma(x)$ is a local symmetry allowed operator involving degrees of freedom from the chains $n, n+1$ immediately above and below the site A , at horizontal position x , with scaling dimension Δ_σ . This implies that the two point functions of these operators obey

$$\langle \mathcal{O}_{n,n+1}^\sigma(x) \mathcal{O}_{n,n+1}^\sigma(x') \rangle = \frac{C_\sigma}{|x - x'|^{2\Delta_\sigma}}, \quad (13)$$

with constants C_σ that are dependent upon the convention for normalizing the fields. The expectation value is taken in the continuum conformal field theory (CFT) describing the decoupled Heisenberg chains in a field. The two point function of two *different* operators $\sigma \neq \sigma'$ vanishes if the operators have different symmetry (or more formally are descended from different primaries in the CFT). The sum over σ may be thought of as being in

order of increasing scaling dimension. Terms with large scaling dimension are strongly irrelevant, and can therefore be neglected.

The first few terms in this expansion are strongly constrained by symmetry. Considering the full microscopic Hamiltonian, the symmetries consist of the lattice space group (*cm**m* - see fig. 2) and the $U(1)$ rotational symmetry of the spins about the z axis. One finds

$$\begin{aligned} \mathcal{H}_2^A = & \lambda_\perp \cos[\theta_n - \theta_{n+1}] + \lambda_z \cos[2\varphi_n - 2\varphi_{n+1}] \quad (14) \\ & + \lambda'_z \partial_x \varphi_n \partial_x \varphi_{n+1} \\ & + \lambda'_\perp \partial_x \theta_n \partial_x \theta_{n+1} \cos[\theta_n - \theta_{n+1}] + \dots \end{aligned}$$

Here we have suppressed the A subscript on n and x on the right hand side to keep the formula compact.

The second order Hamiltonian, \mathcal{H}_2^A , is actually further constrained by an additional symmetry which one obtains only at this order. Specifically, for each term in \mathcal{H}_2^A associated with an inter-chain site A , the chains above and below this site can be *independently* reflected about a vertical axis through site A . This symmetry in fact requires that $\lambda_\perp = 0 + O[(J')^4]$.

A. General prescription for coefficients

In this subsection, we demonstrate that the above effective continuum Hamiltonian is uniquely determined from the microscopic model, by showing how the coefficients (λ_σ) may be obtained in principle from correlation functions of decoupled Heisenberg chains. In the following subsection, we will make some additional simplifications in order to provide explicit expressions.

In general, we can proceed by demanding equality between Eq.(7) and Eq.(14) (Eq.(12)). The coefficients can

then be extracted by taking expectation values of these quantities with microscopic operators whose continuum operator content is known, i.e.

$$\hat{Q}_{j;n,n+1}^\sigma = c_\sigma \mathcal{O}_{n,n+1}^\sigma(x_j) + \dots, \quad (15)$$

where $\hat{Q}_{j;n,n+1}^\sigma$ is a microscopic expression composed of lattice spin operators in the vicinity of site j (horizontal position x') on chains $n, n+1$, and the omitted terms indicated by the ellipses contain only operators of larger scaling dimension than $\mathcal{O}_{n,n+1}^\sigma$. For our problem, the needed \hat{Q}^σ operators and coefficients are

$$\begin{aligned} \frac{1}{2}(S_{j,n}^+ S_{j,n+1}^- + S_{j,n}^- S_{j,n+1}^+) &= b_0^2 \cos[\theta_n - \theta_{n+1}] + \dots \\ S_{j,n}^z S_{j,n+1}^z &= \frac{a_1^2}{2} \cos[2\varphi_n - 2\varphi_{n+1}] + \frac{1}{\pi^2} \partial_x \varphi_n \partial_x \varphi_{n+1} + \dots \\ \frac{1}{2}(S_{j,n}^+ + S_{j+1,n}^+)(S_{j,n+1}^- + S_{j+1,n+1}^-) &+ \text{c.c.} = \\ &= b_0^2 \partial_x \theta_n \partial_x \theta_{n+1} \cos[\theta_n - \theta_{n+1}] + \dots, \quad (16) \end{aligned}$$

where a_1, b_0 are the coefficients in Eqs. (10, 11).

Knowing these, one has

$$\begin{aligned} \langle \hat{Q}_{j;n_A,n_A+1}^\sigma \mathcal{H}_2^A \rangle &= c_\sigma \lambda_\sigma \langle \mathcal{O}_{n_A,n_A+1}^\sigma(x_j) \mathcal{O}_{n_A,n_A+1}^\sigma(x_A) \rangle \\ &= \frac{c_\sigma \lambda_\sigma C_\sigma}{|x_j - x_A|^{2\Delta_\sigma}} + \dots \quad (17) \end{aligned}$$

The neglected terms decay faster with $|x_j - x_A|$, and can therefore be distinguished from the dominant term above. Thus by calculating the left hand side in Eq.(17), and extracting its long-distance behavior, one obtains the coefficient λ_σ , since c_σ, C_σ are presumed known.

From this formulation, we can already deduce the order of the various λ_σ coefficients. Consider first λ'_\perp . We require:

$$\left\langle \left[\frac{(S_{j,n_A}^+ + S_{j+1,n_A}^+)(S_{j,n_A+1}^- + S_{j+1,n_A+1}^-)}{2} + \text{c.c.} \right] \mathcal{H}_2^A \right\rangle = \frac{b_0^2 C'_\perp \lambda'_\perp}{|x_j - x_A|^{2\Delta'_\perp}}. \quad (18)$$

The only obvious constraint to obtain a non-zero result for this correlation function is that the sites k, i in Eq.(7) defining \mathcal{H}_2^A reside on neighboring chains $n_A, n_A + 1$. There are 4 such combinations included in the sum in Eq.(7). Taking the limit $J' \rightarrow 0$ in the resolvent denominator, we obtain

$$\begin{aligned} (\alpha J')^2 S \left\langle \left[\frac{(S_{j,n_A}^+ + S_{j+1,n_A}^+)(S_{j,n_A+1}^- + S_{j+1,n_A+1}^-)}{2} \right] \left[(S_{j,n_A}^- + S_{j+1,n_A}^-) \frac{1}{E_0 - H_0 - h} (S_{j,n_A+1}^+ + S_{j+1,n_A+1}^+) \right] \right\rangle \\ = \frac{b_0^2 C'_\perp \lambda'_\perp}{|x_j - x_A|^{2\Delta'_\perp}}. \quad (19) \end{aligned}$$

There is no reason for the correlation function on the left hand side to vanish. To estimate it, we note that the resolvent denominator is negative for all eigenvalues and bounded below in magnitude by $|E_0 - H_0 - h| > h$. Hence we estimate

$$\lambda'_\perp \lesssim \frac{(\alpha J')^2}{h}, \quad (20)$$

up to $O(1)$ coefficients that can be smooth functions of h/J .

Consider now the remaining two non-zero coefficients, λ_z, λ'_z . Both may be obtained from the expectation value, which we denote G :

$$G = \langle S_{j,n_A}^z S_{j,n_A+1}^z \mathcal{H}_2^A \rangle = G_2 + G_3 + O[(J')^4]. \quad (21)$$

Since the number of particles in chains n_A and $n_A + 1$ must be separately conserved, the sites i, k in the sum in Eq.(7) must be on the same chain to obtain a non-zero result. Hence we may write

$$G \sim (\alpha J')^2 \left\langle S_{j,n_A}^z S_{j,n_A+1}^z [S_{j_A,n_A}^- + S_{j_A+1,n_A}^-] \frac{1}{(E_0 - H_0) - h + J' \sum_{j \text{ nn } A} S_j^z} [S_{j_A,n_A}^+ + S_{j_A+1,n_A}^+] \right\rangle. \quad (22)$$

First consider G_2 , the second order in J' contribution obtained by taking the $J' \rightarrow 0$ limit in the above resolvent. If one takes this limit, then the only remaining operator on chain $n_A + 1$ which is not translationally invariant is the single S_{j,n_A+1}^z . Applying a translation on chain $n_A + 1$, then, one sees that the S_{j,n_A+1}^z can be replaced by the same operator at any other site of the chain, i.e. S_{j',n_A+1}^z , with arbitrary j' . Hence, it can be replaced by average over all sites, i.e. the uniform magnetization, which is a good quantum number and non-fluctuation. Upon making this substitution, one finds

$$\begin{aligned} G_2 &\sim (\alpha J')^2 m_{\text{ch}} \left\langle S_{j,n_A}^z [S_{j_A,n_A}^- + S_{j_A+1,n_A}^-] \frac{1}{(E_0 - H_0) - h} [S_{j_A,n_A}^+ + S_{j_A+1,n_A}^+] \right\rangle \\ &\lesssim -\frac{(\alpha J')^2}{h} m_{\text{ch}} \langle S_{j,n_A}^z [S_{j_A,n_A}^- + S_{j_A+1,n_A}^-] [S_{j_A,n_A}^+ + S_{j_A+1,n_A}^+] \rangle, \end{aligned} \quad (23)$$

where the \lesssim is to be understood as indicating an inequality of magnitude (not sign), using the same bound on the denominator as before. The final line is straightforwardly analyzed from known results for Heisenberg chains. Because there is only a single field (S_{j,n_A}^z) at position x_j , the long distance decay of G_2 contains two pieces: an *oscillatory* term decaying with the exponent Δ_z and a non-oscillatory term decaying with the exponent Δ'_z . The oscillatory term indicates a vanishing contribution in the continuum limit. The non-oscillating term has *half* the expected exponent, $2\Delta'_z$ to correspond to a contribution to λ'_z . It instead represents the generation of a term proportional to $\partial_z \varphi_n$, which generates a small smooth renormalization of the magnetization curve but is otherwise redundant. Thus *neither* λ_z nor λ'_z are generated at $O[(J')^2]$.

In order to calculate the leading contribution to λ_z and λ'_z then, one must calculate G_3 by expanding the resolvent in Eq.(22) to first order in J' , giving

$$\begin{aligned} G_3 &\sim -J' (\alpha J')^2 \left\langle S_{j,n_A}^z S_{j,n_A+1}^z [S_{j_A,n_A}^- + S_{j_A+1,n_A}^-] \frac{1}{(E_0 - H_0) - h} \left[\sum_{k \text{ nn } A} S_k^z \right] \frac{1}{(E_0 - H_0) - h} [S_{j_A,n_A}^+ + S_{j_A+1,n_A}^+] \right\rangle \\ &\lesssim -\frac{J' (\alpha J')^2}{h^2} \left\langle S_{j,n_A}^z S_{j,n_A+1}^z [S_{j_A,n_A}^- + S_{j_A+1,n_A}^-] \left[\sum_{k \text{ nn } A} S_k^z \right] [S_{j_A,n_A}^+ + S_{j_A+1,n_A}^+] \right\rangle \\ &\sim \frac{a_1^2}{2} \frac{\lambda_z C_z}{|x_j - x_A|^{2\Delta_z}} + \frac{1}{\pi^2} \frac{\lambda'_z C'_z}{|x_j - x_A|^{2\Delta'_z}} \end{aligned} \quad (24)$$

and so we find that λ_z and λ'_z are generally of order $(J')^3$.

B. Explicit Calculation for Spin 1/2 System

The above results provide a general formulation to calculate the coefficients in the effective Hamiltonian, Eq.(14), from correlation functions of a single Heisenberg chain in a field. Because of the complicated form of \mathcal{H}_2^A , however, even these one-dimensional correlation functions are not simple to obtain analytically, or extract from known results. In this subsection, therefore, we make some additional simplifying assumptions which allow an explicit calculation of the effective Hamiltonian for the spin 1/2 system. This calculation confirms the

general structure obtained above and allows for some semi-quantitative estimates.

Consider again the the second order (in α) contribution to effective Hamiltonian density

$$\mathcal{H}_2^A = \left(\frac{\alpha J'}{2} \right)^2 \sum_{\langle ik; A \rangle} S_k^- \frac{1}{(E_0 - H_0) - h + J' \sum_{j \text{ nn } A} S_j^z} S_i^+.$$

When the operator above acts within the low energy subspace, one would like to say that both the term proportional to J' and the $(E_0 - H_0)$ term in the denominator are small, but it is not clear that $(E_0 - H_0)$ is small

unless it acts *directly* on the low energy states. So, we exchange the S_i^+ operator and the resolvent using spin 1/2 anticommutation relations to obtain

$$\mathcal{H}_2^A = \left(\frac{\alpha J'}{2}\right)^2 \sum_{\langle ik;A \rangle} S_k^- \left[S_i^x R_{Ai;y} + i S_i^y R_{Ai;x} \right].$$

where

$$R_{Ai;\sigma} = \left[(E_0 - H_0) - h + J' \sum_{j \text{ nn } A} S_j^z - 2h S_i^z + 2J \sum_{j \text{ nni}} (S_i^z S_j^z + S_i^\sigma S_j^\sigma) \right]^{-1}.$$

However, to be able to expand the $R_{Ai;\sigma}$ operators, we must make two rather unphysical assumptions. First, we take $J \ll h$ and second, we only let the magnetic field couple to I spins. This second assumption may not be too bad, however, since our eventual replacement of the chains of S spins with Luttinger liquids takes the effect of the external field into account via the Luttinger parameter.

After making the approximations just discussed and expanding, we finally obtain the following leading contributions:

$$\begin{aligned} \mathcal{H}_2^A \approx & -\frac{(\alpha J')^2}{4h} \sum_{\langle ik;A \rangle} S_k^- S_i^+ \\ & -\frac{(\alpha J')^2 J'}{4h^2} \sum_{\langle ij;A \rangle} S_k^- S_i^+ S_j^z \\ & +\frac{(\alpha J')^2 J}{4h^2} \sum_{\langle ik;A \rangle} \sum_{j \text{ nni}} S_k^- S_i^+ S_j^z - S_k^- S_i^z S_j^+ . \end{aligned} \quad (25)$$

In what follows, we set α back to 1.

To connect this result to the continuum limit of the interchain coupling Eq.(14), we now proceed to bosonize the spin operators through the identifications in Eq.(10) and Eq.(11). Keeping only contributions that involve products of operators from both chains, we find from the first term

$$\begin{aligned} \sum_{\langle ik;A \rangle} S_k^- S_i^+ &= (S_{j_A, n_A}^- + S_{j_A+1, n_A}^-)(S_{j_A, n_A+1}^+ + S_{j_A+1, n_A+1}^+) \\ &+ [n_A \leftrightarrow n_A + 1] \\ &= 2b_0^2 \partial_x \theta_n \partial_x \theta_{n+1} \cos[\theta_n - \theta_{n+1}] + \dots \end{aligned}$$

which confirms that λ'_\perp is of order J'^2 , as we already argued by symmetry.

The second term in Eq.(25) must be treated with more care, however. Now, we only need to keep operators that conserve particle number on each chain, since we have

already found the leading contribution to λ'_\perp and since $\lambda_\perp = O[(J')^4]$ by symmetry. After also dropping operators that do not have contributions from both chains, we find

$$\begin{aligned} \sum_{\langle ikj;A \rangle} S_k^- S_i^+ S_j^z &= (S_{j_A, n_A}^- + S_{j_A+1, n_A}^-)(S_{j_A, n_A+1}^+ + S_{j_A+1, n_A+1}^+) \\ &\times (S_{j_A, n_A+1}^z + S_{j_A+1, n_A+1}^z) + [n_A \leftrightarrow n_A + 1] \\ &= (S_{j_A, n_A+1}^z + S_{j_A+1, n_A+1}^z)(1 - S_{j_A, n_A}^z - S_{j_A+1, n_A}^z \\ &+ \{S_{j_A, n_A}^- S_{j_A+1, n_A}^+ + S_{j_A, n_A}^+ S_{j_A+1, n_A}^-\}) + [n_A \leftrightarrow n_A + 1]. \end{aligned}$$

It is the last operator in curly braces (the dimerization operator) that must be bosonized carefully. Since it involves products of operators at very short distances, all subleading bosonic operators omitted in Eq.(10) must be resummed, leading to the following continuum limit²³

$$(S_{j,n}^- S_{j+1,n}^+ + S_{j,n}^+ S_{j+1,n}^-) \sim c_0 + \frac{c'_0}{\pi} \partial_x \varphi_n - c_1 \cos[2\varphi_n + Qx]. \quad (26)$$

After using the above identity and collecting all relevant terms, we find that

$$\begin{aligned} \sum_{\langle ikj;A \rangle} S_k^- S_i^+ S_j^z &= \frac{2}{\pi^2} (c'_0 - 4) \partial_x \varphi_{n_A} \partial_x \varphi_{n_A+1} \\ &- (2a_1^2 + a_1 c_1)(1 + \cos Q) \cos[2\varphi_{n_A} - 2\varphi_{n_A+1}] + \dots \end{aligned} \quad (27)$$

demonstrating that λ_z and λ'_z are both of order J'^3 . The last term in Eq.(25) does not generate any relevant interactions in the bosonic theory.

To summarize, then, we have explicitly calculated the couplings in the effective Hamiltonian Eq.(14) for the spin 1/2 system up to third order in J' and the results confirm the predictions of our symmetry arguments (which apply to systems with any spin). The explicit couplings are

$$\begin{aligned} \lambda_\perp &= 0 \\ \lambda'_\perp &= \frac{-J'^2 b_0^2}{2h} \\ \lambda_z &= \frac{J'^3}{2h^2} (2a_1^2 + a_1 c_1)(1 + \cos Q) \\ \lambda'_z &= \frac{J'^3}{\pi^2 h^2} (4 - c'_0). \end{aligned} \quad (28)$$

IV. ANALYSIS OF THE CONTINUUM MODEL

We have arrived, then, at a low energy effective theory describing the Heisenberg antiferromagnet on the anisotropic kagome lattice that takes the general form

$$\begin{aligned} H_{\text{eff}} &= \sum_n \int_x \left[\frac{v}{2\pi} \frac{1}{g} (\partial_x \varphi_n)^2 + \frac{vg}{2\pi} (\partial_x \theta_n)^2 + \lambda'_z \partial_x \varphi_n \partial_x \varphi_{n+1} \right. \\ &+ \lambda_\perp \cos[\theta_n - \theta_{n+1}] + \lambda_z \cos[2\varphi_n - 2\varphi_{n+1}] \\ &\left. + \lambda'_\perp \partial_x \theta_n \partial_x \theta_{n+1} \cos[\theta_n - \theta_{n+1}] + \dots \right]. \end{aligned} \quad (29)$$

A. RG Analysis

To better understand the behavior of the above model at large length scales, we employ the momentum space RG procedure, lowering the cutoff from $\Lambda \rightarrow \Lambda - d\Lambda$ and defining $dl = d\Lambda/\Lambda$. Including the marginal term $\lambda'_z \partial_x \varphi_n \partial_x \varphi_{n+1}$ in the fixed-point action and expanding all quantities to first order in λ'_z leads to the following tree-level flow equations for the other couplings (note that λ'_z may be regarded as a constant independent of the scale of the effective theory):

$$\begin{aligned} \frac{d\lambda'_\perp}{dl} &= \left(-\frac{1}{2g} + \frac{\pi\lambda'_z}{4v} \right) \lambda'_\perp \\ \frac{d\lambda_\perp}{dl} &= \left(2 - \frac{1}{2g} + \frac{\pi\lambda'_z}{4v} \right) \lambda_\perp + \frac{\pi\Lambda^2\lambda'_z}{16v} \lambda'_\perp \\ \frac{d\lambda_z}{dl} &= \left(2 - 2g - \frac{\pi g^2 \lambda'_z}{v} \right) \lambda_z. \end{aligned} \quad (30)$$

These flow equations imply that for $\lambda'_z \neq 0$ and $\lambda'_\perp \neq 0$, a finite value of λ_\perp will be generated in the long-wavelength theory. Also, one can see that the θ -interaction (with coupling λ_\perp) is relevant for $g \gtrsim \frac{1}{4} - \frac{\pi\lambda'_z}{32v}$ while the φ -interaction (with coupling λ_z) is relevant for $g \lesssim 1 - \frac{\pi\lambda'_z}{2v}$.

Now for the case of spin 1 chain, F  th has determined numerically that g is always greater than one (see F  th²⁴, fig. 5). In particular, g is one at a chain magnetization per site of $m_{\text{ch}} = 0$, reaches a maximum value of $g \approx 1.46$ at $m_{\text{ch}} \approx 0.36$ and approaches one again in the limit of full polarization. So, for a spin 1 system described by (29), the θ -interaction is always relevant, while the φ -interaction is always irrelevant.

On the other hand, for a spin 1/2 chain, the exact Bethe ansatz solution shows that g increases nearly linearly from 1/2 to 1 as the chain magnetization per site increases from $m_{\text{ch}} = 0$ to $m_{\text{ch}} = 1/2$ (see Hikihara and Furusaki²⁵, fig. 2). So for a spin 1/2 system described by (29), the φ -interaction and the θ -interaction are nearly always relevant, and we must find a further criterion to determine which interaction dominates the behavior of the system. So, we solve the flow equations and determine which coupling constant becomes ~ 1 first as one continues to integrate out high momentum modes.

B. Competing Interactions in the Spin 1/2 System

We take the initial conditions given by Eq.(28). Then, defining

$$\begin{aligned} \beta'_\perp &= \left(-\frac{1}{2g} + \frac{\pi\lambda'_z}{4v} \right), \\ \beta_\perp &= \left(2 - \frac{1}{2g} + \frac{\pi\lambda'_z}{4v} \right) \\ \beta_z &= \left(2 - 2g - \frac{\pi\lambda'_z g^2}{v} \right) \end{aligned}$$

the solutions to the flow equations for λ_\perp and λ_z are

$$\begin{aligned} \lambda_\perp(l) &= \frac{J'^5}{h^3} \frac{\Lambda^2 b_0^2}{64\pi v} (4 - c'_0) (e^{\beta_\perp l} - e^{\beta'_\perp l}) \\ &\approx \frac{J'^5}{h^3} \frac{\Lambda^2 b_0^2}{64\pi v} (4 - c'_0) e^{\beta_\perp l} \\ \lambda_z(l) &= \frac{J'^3}{2h^3} (a_1 c_1 + 2a_1^2) (1 + \cos Q) e^{\beta_z l}. \end{aligned}$$

We can ignore the term $\sim e^{\beta'_\perp l}$ in $\lambda_\perp(l)$ since it rapidly goes to zero unless $g \gtrsim 1/\lambda'_z$. Identifying e^l as L/a , i.e. the ratio of the renormalized cutoff length scale to the bare lattice scale, we find that

$$\begin{aligned} \lambda_\perp(l) &\sim J'^5 \left(\frac{L}{a} \right)^{\beta_\perp} \\ \lambda_z(l) &\sim J'^3 \left(\frac{L}{a} \right)^{\beta_z} \end{aligned}$$

If we then define L_\perp and L_z as the length scales at which the θ and φ couplings, respectively, become ~ 1 , we find that

$$\frac{L_z}{L_\perp} \sim (J')^{5/\beta_\perp - 3/\beta_z}$$

The exponent is positive for $0 < g < g_c$ and negative for $g_c < g \lesssim 1$, where $g_c \approx 0.636 - 0.635 \frac{\lambda'_z}{v} + \dots$. This means that for $g < g_c$, $L_z \ll L_\perp$ and the φ interaction is the dominant one. For $g > g_c$, the θ interaction is dominant.

C. Ordered Phases in the Gaussian Approximation

1. Ferrimagnetic Phase

First, consider a system described by the Hamiltonian (29) in the limit where $g < g_c$, that is a spin 1/2 system with magnetization per site between $m = 0$ and $m \approx 1/5$ (on a scale where full polarization is $m = 1/2$). In this limit, the dominant relevant coupling is the φ -interaction with coupling λ_z , so we drop all other couplings not in the fixed-point action. The θ fields may then be integrated out to yield an action entirely in terms of the φ fields:

$$\begin{aligned} S \approx \sum_n \int_\tau \int_x \frac{v}{2\pi g} \left[(\partial_x \varphi_n)^2 + \frac{1}{v^2} (\partial_\tau \varphi_n)^2 \right] \\ + \lambda'_z \partial_x \varphi_n \partial_x \varphi_{n+1} + \lambda_z \cos[2\varphi_n - 2\varphi_{n+1}]. \end{aligned} \quad (31)$$

Since the interaction term in the above theory is relevant, the long-wavelength modes of the φ fields will fluctuate very little from chain to chain if the coupling λ_z is negative, while they will differ by $\pi/2$ from chain to chain if the coupling is positive. If λ_z is positive, then, redefine the φ fields by $\varphi_n(x) \rightarrow \varphi_n(x) + \frac{\pi}{2}n$. The coupling term for these new fields will now be negative.

After the appropriate redefinition, fluctuations of the φ fields from chain to chain will be suppressed at zero

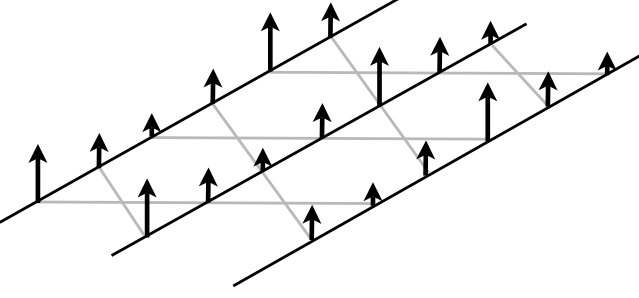


FIG. 3: An example of a ferrimagnetically ordered ground state of the spin 1/2 system with magnetization $m \lesssim 1/5$ and $\lambda_z < 0$.

temperature, and so it is reasonable to expand the cosine term in the action to second order about zero. Upon performing a fourier transform in the chain index n , the resulting gaussian action is

$$S \approx \frac{v}{2\pi g} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{x,\tau} \left[\left(1 - \frac{2\pi g}{v} \lambda'_z \cos k\right) |\partial_x \varphi_k|^2 + \frac{1}{v^2} |\partial_\tau \varphi_k|^2 + m_\varphi^2(k) v^2 |\varphi_k|^2 \right] \quad (32)$$

where the effective mass m_φ is defined as

$$m_\varphi^2(k) = \frac{8\pi g}{v^3} (1 - \cos k) \lambda_z. \quad (33)$$

A system described by the action (32) has excitations with a dispersion relation

$$\omega_\varphi^2(\vec{p}, k) = \left(1 - \frac{2\pi g}{v} \lambda'_z \cos k\right) p^2 v^2 + m_\varphi^2(k) v^4 \quad (34)$$

which vanishes as $\vec{p}, k \rightarrow 0$, implying the existence of gapless excitations. Indeed, the action (32) is invariant under the transformation

$$\varphi_{k=0}(x) \rightarrow \varphi_{k=0}(x) + \bar{\varphi} \quad (\bar{\varphi} \text{ constant})$$

because of the invariance of the system under translations along the chain or x -direction. The gapless excitations are the Goldstone modes associated with the breaking of this symmetry.

Restoring the original definition of the fields $\varphi_n(x)$ as appropriate, we see that for small enough g , the φ_n fields will be ordered as $\langle \varphi_n(x) \rangle = \bar{\varphi} + \frac{\pi}{2}n$ if λ_z is positive and $\langle \varphi_n(x) \rangle = \bar{\varphi}$ if λ_z is negative. Then, from Eq.(11) we see that the spins will be ordered ferrimagnetically:

$$\langle S_n^z(x) \rangle \sim \begin{cases} m_{\text{ch}} + \delta m (-1)^n \cos(2\bar{\varphi} + Qx) & \text{if } \lambda_z > 0 \\ m_{\text{ch}} + \delta m \cos(2\bar{\varphi} + Qx) & \text{if } \lambda_z < 0. \end{cases} \quad (35)$$

(see fig. 3 for an illustration). The amplitude of the ferrimagnetic spin density wave oscillation, $\delta m \sim$

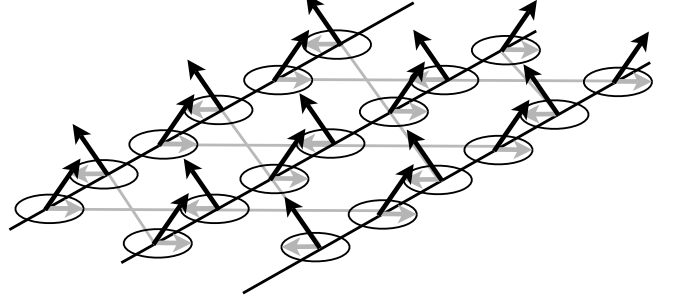


FIG. 4: The XY antiferromagnetically ordered ground state that occurs for both the spin 1/2 and spin 1 systems in a large enough field ($\lambda_\perp < 0$ case shown).

$a_1(J'/J)^{3g/(2-2g)}$ is small for small J' , due to the fact that $S_n^z(x)$ is a continuum operator with all fluctuations on length scales below L_z integrated out. Since $g < 1$ in the spin 1/2 model, this factor will tend to suppress the oscillations in the magnetization.

2. XY Antiferromagnetic Phase

Now, consider a system described by the Hamiltonian (29) but in the limit where $g > g_c$, namely a spin 1/2 system with magnetization per site $m \gtrsim 1/5$ or a spin 1 system in a sufficiently large field (larger than both the Haldane gap and J'). In this limit, the dominant relevant coupling is the θ -interaction with coupling λ_\perp , so we drop all other couplings not in the fixed-point action.

We proceed much as before, integrating out the φ fields and redefining the θ fields if λ_\perp is positive. In this case, however, the appropriate redefinition is $\theta_n(x) \rightarrow \theta_n(x) + \pi n$. Once the fields are defined such that the cosine interaction makes them slowly varying at zero temperature, we expand the cosine to obtain a gaussian theory once again, but with an effective mass

$$m_\theta^2(k) = \frac{2\pi}{gv^3} (1 - \cos k) \lambda_\perp \quad (36)$$

and a dispersion

$$\omega_\theta^2(p, k) = \left(1 - \frac{2\pi g}{v} \lambda'_z \cos k\right) [p^2 v^2 + m_\theta^2(k) v^4]. \quad (37)$$

The gapless modes in this limit are the Goldstone modes associated with the breaking of the shift symmetry

$$\theta_{k=0}(x) \rightarrow \theta_{k=0}(x) + \bar{\theta} \quad (\bar{\theta} \text{ constant})$$

associated with the global rotational symmetry of all spins about the magnetic field direction.

Restoring the original fields, we expect that, for large enough g , $\langle \theta_n(x) \rangle = \bar{\theta} + \pi n$ if $\lambda_\perp > 0$ or $\langle \theta_n(x) \rangle = \bar{\theta}$ if $\lambda_\perp < 0$. From the identification (10), we therefore expect

an XY antiferromagnetic ordering of the spins:

$$\langle S_{j,n}^+ \rangle \sim \begin{cases} (-1)^{j+n} m_s e^{i\bar{\theta}} & \text{if } \lambda_{\perp} > 0 \\ (-1)^j m_s e^{i\bar{\theta}} & \text{if } \lambda_{\perp} < 0 \end{cases}, \quad (38)$$

where $m_s \sim b_0(J'/J)^{5/(8g-2)}$ is the staggered magnetization, which is reduced by fluctuations below the scale L_{\perp} . See fig. 4 for an illustration.

D. Lattice Effects in the Spin 1/2 Ferrimagnetic Phase

1. Commensurate Ferrimagnetic Order

For a spin 1/2 system with $g < g_c$, the ordering of the zero mode of the φ field corresponds to the breaking of the translational symmetry of the system in the x direction (along the chains). However, this symmetry is not actually continuous in the microscopic theory, but reduces to the discrete translational symmetry of the lattice.

In order to reconcile the continuum and microscopic theories then, we observe that the Hamiltonian (29) may also contain terms of the form $\cos[2p\varphi_n(x)]$ for certain values of p that must be determined by symmetry.

Under translation by one lattice spacing along the chain direction, the φ fields transform as

$$\varphi_n(x) \rightarrow \varphi_n(x-1) - Q/2. \quad (39)$$

So, for a term of the form $\cos[2p\varphi_n(x)]$ to respect the translational symmetry of the system, it must be that $pQ = 2\pi k$, where k is an integer. Moreover, the action should be invariant under $\varphi_n(x) \rightarrow \varphi_n(x) + \pi$, so k must be chosen to make p an integer.

Recalling that $Q = 2\pi(m_{\text{ch}} + 1/2)$, both requirements imply that only the set $\{p_j\}$ are allowed, where

$$p_j = \frac{k_j}{m_{\text{ch}} + \frac{1}{2}} \quad (40)$$

and k_j is the j th positive integer (in increasing order) such that p_j is an integer. We may identify

$$\bar{\lambda} = \frac{1}{m_{\text{ch}} + \frac{1}{2}} = \frac{2\pi}{Q} \quad (41)$$

as the wavelength of the (continuum) ferrimagnetic wave (35). But, for a general magnetization, the wavelength $\bar{\lambda}$ will not be an integer multiple of the lattice spacing. Instead, the ferrimagnetic wave above, when restricted to values of x that coincide with the lattice, will appear to be periodic with an effective wavelength $\lambda_{\text{eff}} = k_0 \bar{\lambda} = p_0$.

So then, given that for a fixed magnetization the corresponding term $\cos[2p_0\varphi_n]$ will appear in the Hamiltonian, it will bring the discrete translational symmetry of the lattice to bear on the ordered phase by restricting the values of $\langle \varphi_n(x) \rangle = \bar{\varphi}$ to coincide with its minima. For

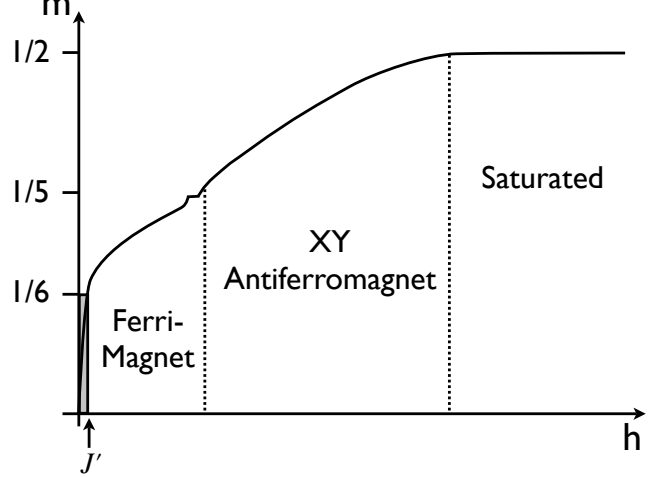


FIG. 5: Schematic graph of the magnetization curve and phases of the spin-1/2 system. Note the hypothetical plateau just below the crossover point. The shaded region indicates the parameter space where our model does not apply.

a negative coefficient of $\cos[2p_0\varphi_n]$, this will force the ferrimagnetic oscillations to be symmetric about a particular lattice site. On the other hand, if the coefficient is positive the oscillations will be symmetric about the point on the lattice between two particular neighboring sites.

2. Magnetization Plateaux

In addition to pinning the ferrimagnetic order to the lattice, the symmetry-allowed $\cos[2p\varphi_n(x)]$ terms in the Hamiltonian may lead to the appearance of plateaux in the magnetization curve of the spin 1/2 system.

For general (but small) values of the external field, the ground state of the spin 1/2 system will be, in bosonic terms, a state in which the $\varphi_n(x)$ fields on each chain fluctuate about some constant average value as we discussed in detail above (we take the coupling λ_z to be negative in this section for simplicity - the other case follows straightforwardly). In such states, the chain magnetization per site smoothly oscillates as a function of x about its average (to zeroth order in J') value m_{ch} . Note that m_{ch} is simply the magnetization of a decoupled Heisenberg chain and as such, is a monotonically increasing function of the external magnetic field.²⁵

Now consider some particular magnetization m^P (which corresponds to a chain magnetization $m_{\text{ch}}^P = 3m^P/2 - 1/4$) and first assume that the external field has been tuned to a value h^P such that $m_{\text{ch}}(h^P) = m_{\text{ch}}^P$. In such an external field, the most important symmetry allowed term of the type $\cos[2p\varphi_n(x)]$ that may be added to the Hamiltonian is the one with the smallest value of

p given by Eq.(40), namely

$$p_0 = \frac{k_0}{m_{\text{ch}}^P + \frac{1}{2}}, \quad (42)$$

where k_0 is defined as the smallest positive integer such that p_0 is an integer.

To illustrate, then, how a plateau may arise at this magnetization, we consider the Hamiltonian

$$H_{\text{eff}} = \sum_n \int_x \frac{v}{2\pi g} \left[(\partial_x \varphi_n)^2 + \frac{1}{v^2} (\partial_\tau \varphi_n)^2 \right] + \lambda'_z \partial_x \varphi_n \partial_x \varphi_{n+1} - |\lambda_{z0}| \cos[2(\varphi_n - \varphi_{n+1})] - \eta_0 \cos[2p_0 \varphi_n + \delta x] + \dots \quad (43)$$

and no longer think of the external field as fixed, but as varying in some small range about h^P . Since the coefficient δ is zero for $h = h^P$ by the definition of p_0 , symmetry considerations show that it must be given in general by

$$\delta(h) = 2\pi p_0 [m_{\text{ch}}(h) - m_{\text{ch}}^P]. \quad (44)$$

Now, if we integrate out the short wavelength modes of the system in an iterative RG procedure (but *without* rescaling the high momentum cutoff), the effective coupling λ_z will vary as

$$\lambda_z[L] \sim \lambda_{z0} (L)^{-2g} \quad (45)$$

until we reach a length scale L_z defined such that $\lambda_z[L_z] \sim 1/(L_z)^2$ (i.e. this is the scale such that λ_z would be $O(1)$ if we were rescaling at every RG step). Therefore, we find that

$$L_z \sim (\lambda_{z0})^{-1/(2-2g)}. \quad (46)$$

At this scale, the chains will be strongly coupled and the φ fields will vary slowly from chain to chain.

Then, since the φ fields vary slowly from chain to chain, it is reasonable to take a continuum limit $\varphi_n(x) \rightarrow \varphi(x, y)/\sqrt{d}$ (where d is the chain spacing). Within such an effective $(2+1)$ -dimensional theory, the long-wavelength fluctuations of the field $\varphi(x, y)$ are bounded and so the term in the action $\eta_0 \cos[2p_0 \varphi]$, which may have been an irrelevant perturbation to a decoupled chain is now a relevant perturbation, regardless of the value of p_0 .

Now, since the scaling dimension of the $\cos[2p_0 \varphi]$ term is $p_0^2 g$, the effective coupling η becomes

$$\eta(L_z) \sim \eta_0 \cdot (L_z)^{-p_0^2 g} \sim \eta_0 \cdot (\lambda_{z0})^{\frac{p_0^2 g}{2-2g}} \quad (47)$$

as the momentum cutoff is lowered to $1/L_z$ under the RG procedure. Since λ_{z0} is small (generally $\sim J'^3$) and $g < 1$ for the spin 1/2 system in its ferrimagnetic phase, this renormalized value of η will decrease rapidly as a function of p_0 .

Finally, then, we determine if a plateau develops around the magnetization m^P by considering the competition between the kinetic energy terms in H_{eff} and the term with coupling η . The latter term is minimized if $\langle \varphi_n(x) \rangle = \frac{-\delta(h)}{2p_0} x$ (that is, if $\langle S_n^z(x) \rangle = m_{\text{ch}}^P$ regardless of the external field - a plateau). However, such a field configuration costs some kinetic energy and the resulting total change in the linear energy density is

$$\Delta E_n(x) \sim \frac{v}{2\pi g} \left(\frac{\delta(h)}{2p_0} \right)^2 - \eta = \frac{v [\delta(h)]^2}{8\pi g p_0^2} - \eta. \quad (48)$$

Therefore, we see that this arrangement remains favorable (i.e. the plateau exists) as long as

$$|\delta(h)| < \sqrt{\eta} 2p_0 \sqrt{\frac{2\pi g}{v}} \quad (49)$$

$$\implies |m_{\text{ch}}(h) - m_{\text{ch}}^P| \lesssim (J')^{3p_0^2 g/4(1-g)}.$$

A significant plateau will only appear, then, for a relatively small p_0 , which is itself determined by m_{ch}^P through (42).

We therefore expect the widest plateau, if one appears at all, to appear just below the crossover point from the ferrimagnetic to the XY antiferromagnetic phase in the spin 1/2 system (see fig. 5). Furthermore, we may estimate the dependence of the plateau width on J' from our earlier estimate that $g_c \sim 0.636$ and from the fact that the smallest p_0 allowed by Eq.(42) near the crossover is about 9. Using these values in Eq.(50), the power of J' is ~ 100 , so any plateau near the crossover will be narrow indeed.

Acknowledgments

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